Speirits, F.C., and Barnett, S.M. (2013) Do waves carrying orbital angular momentum possess azimuthal linear momentum? Physical Review Letters, 111 (10). Art. 103602. ISSN 0031-9007

Do Waves Carrying Orbital Angular Momentum Possess Azimuthal Linear Momentum?

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(Received 21 May 2013; published 4 September 2013)

All beams are a superposition of plane waves, which carry linear momentum in the direction of propagation with no net azimuthal component. However, plane waves incident on a hologram can produce a vortex beam carrying orbital angular momentum that seems to require an azimuthal linear momentum, which presents a paradox. We resolve this by showing that the azimuthal momentum is not a true linear momentum but the azimuthal momentum density *is* a true component of the linear momentum density.

DOI: 10.1103/PhysRevLett.111.103602 PACS numbers: 42.50.Tx

Vortices carrying angular momentum occur both naturally and can be created experimentally in many different media [1]. Twisted ultrasound waves [2,3], vortices in Bose-Einstein condensates [4,5], quantum turbulence in superfluid helium [6], Alfvén vortices in plasmas [7], and electron vortices in electron microscopes [8–10] all point to the ubiquity of this phenomenon and highlight the growing field of interest of both studying and utilizing these phase singularities.

Vortices in monochromatic electromagnetic fields have become increasingly widely studied across a large range of wavelengths, with many experimental uses being identified, including optical tweezers [11], spanners [12], and potential astronomical [13] and communication applications [14]. There is also a growing need to understand the behavior of the momentum and associated wave vector around a beam's central singularity in order to fully exploit the emerging field of metamaterials, for example, the subdiffraction level imaging that is now possible with hyperlenses [15].

We have come to associate vortices in optical fields with the presence of orbital angular momentum (OAM). The easiest way to picture this is to reflect that there is necessarily a phase gradient around a vortex and, hence, a flow of azimuthal momentum in the region around the vortex line [16]. If we multiply this local azimuthal momentum by the distance from the vortex we arrive at an orbital angular momentum along the vortex. It is in this way that the idea of OAM was introduced into optics [17]. However, it is natural to ask whether this azimuthal momentum is a true linear momentum or not. We find that it is not.

For many wave phenomena it suffices to consider a complex scalar field ψ. The momentum density can then be written in terms of the gradient of this field in the form

$$\mathbf{P} = \Im(\psi^* \nabla \psi), \tag{1}$$

where $\Im$ represents the imaginary part and a constant of proportionality may be accounted for in the normalization of ψ. This procedure arises in the flow of probability in quantum theory [18,19], in the theory of quantum fluids [20,21], and also in optics, where the Poynting vector for a single polarization may be written in this way in the eikonal approximation [22].

Vortices in the field manifest as phase singularities where the field takes the value zero and there is an accumulation of phase $2\pi l$ on traversing a closed path around the vortex. Here l is the charge of the vortex and may take any integer value. There is a flow associated with this phase change, which appears in the momentum density. In particular, if the vortex lies along the z axis then the azimuthal component of the momentum density is

$$P_\phi = \Im\left(\psi^* \frac{1}{\rho}\frac{\partial}{\partial\phi}\psi\right), \tag{2}$$

where we have introduced the cylindrical polar coordinates (ρ, ϕ, z). We can obtain the density of orbital angular momentum about the vortex simply by multiplying this by the local distance ρ from the vortex:

$$L_z = \Im\left(\psi^* \frac{\partial}{\partial\phi}\psi\right). \tag{3}$$

Note that, for these scalar waves, the existence of an angular momentum density *requires* a local density of azimuthal linear momentum.

It is straightforward to find both the total azimuthal momentum and the total angular momentum by integrating these densities over all space. The simplest case to consider is that of a single vortex line along the z axis, for which our field takes the form

$$\psi = e^{il\phi}u(\rho, z, t). \tag{4}$$

This gives a total azimuthal momentum and total angular momentum of the forms

$$p_\phi = l \int dV \frac{|u|^2}{\rho}, \qquad \ell = l \int dV |u|^2, \tag{5}$$

both of which are positive for positive values of l (the analogous case for acoustic waves is addressed in [23]). It is clear that a positive value of the total orbital angular momentum requires also a positive value of the total azimuthal momentum. At this point it is necessary to emphasize the difference between the total azimuthal momentum p_ϕ, which is clearly nonzero, and the azimuthal component of the total linear momentum, which is identically zero. Indeed, it is intuitive that the azimuthal component of the total linear momentum must be zero as integrating over all space will average out the azimuthal contributions of the local momentum densities, leaving a component of the linear momentum solely along the direction of propagation z [24].

There is, however, a paradox associated with the apparently simple observation that a vortex beam has a nonzero total azimuthal momentum: all waves of well-defined linear momentum have a *zero* azimuthal component of momentum. The plane waves, with spatial dependence $e^{i\mathbf{k}\cdot\mathbf{r}}$, are associated with precisely defined momentum, the value of which is proportional to the wave vector $\mathbf{k}$. However, no azimuthal component of this wave vector appears in the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i(k_\rho \rho + k_z z)}. \tag{6}$$

This is physically reasonable, of course, as the plane wave, being spatially homogeneous, has no preferred axis about which to define an azimuthal component. However, it does present a paradox. How can a wave with a nonzero azimuthal momentum arise from a superposition of plane waves, each of which has zero azimuthal momentum? In quantum mechanics, for example, it is wholly remarkable to find a nonzero expectation value of a quantity for a state that is a superposition of component states each of which has a zero expectation value for that quantity.

In order to resolve this paradox, it is helpful to consider the azimuthal momentum as a quantum operator by writing it in the form

$$\hat{p}_\phi = -i\frac{\hbar}{\rho}\frac{\partial}{\partial\phi} = \frac{1}{\rho}\hat{\ell}_z, \tag{7}$$

where $\hat{\ell}_z$ is the z component of the orbital angular momentum operator. It is then straightforward to show that this operator does not commute with the x and y components of the linear momentum:

$$[\hat{p}_x, \hat{p}_\phi] = -\hbar^2 \frac{\sin\phi}{\rho}\frac{\partial}{\partial\rho}, \qquad [\hat{p}_y, \hat{p}_\phi] = \hbar^2 \frac{\cos\phi}{\rho}\frac{\partial}{\partial\rho}. \tag{8}$$

We must conclude that p_ϕ is not strictly a linear momentum. The radial momentum operator $\hat{p}_\rho$, although it is difficult to define [25], is also not strictly a linear momentum and does not commute with $\hat{p}_x$ and $\hat{p}_y$.

To further illustrate this point we recall that the eigenstates of linear momentum are the plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}$ but the eigenstates of our azimuthal momentum operator take the form

$$u(l/\rho_0) = e^{il\phi}\delta(\rho - \rho_0), \tag{9}$$

where the corresponding azimuthal momentum eigenvalue is $\hbar l/\rho_0$.

The eigenstates of linear momentum are invariant on propagation and this of course reflects the conservation of linear momentum. Physically, linear momentum is a manifestation of homogeneity: there are no preferred positions. It is clear, however, that the azimuthal momentum must be defined relative to an axis and an eigenstate of azimuthal momentum is not invariant as diffraction will cause a spreading of the radial field distribution. It follows that the azimuthal momentum will not be a conserved quantity. The relevant conserved quantity is the orbital angular momentum, which is $\hat{\ell}_z = \rho\hat{p}_\phi$.

While the azimuthal momentum is not a true linear momentum, there is a simple relationship between the *densities* of linear momentum and of azimuthal momentum. We can define an operator for the density of linear momentum in the form [20]

$$\hat{\mathbf{p}}(\mathbf{R}) = \frac{1}{2}[\hat{\mathbf{p}}\delta(\hat{\mathbf{r}} - \mathbf{R}) + \delta(\hat{\mathbf{r}} - \mathbf{R})\hat{\mathbf{p}}]. \tag{10}$$

The azimuthal momentum density is simply the component of this operator in the azimuthal direction:

$$\hat{p}_\phi = \frac{1}{2}\{\cos\phi\,\hat{p}_y - \sin\phi\,\hat{p}_x, \delta(\hat{\mathbf{r}} - \mathbf{R})\}. \tag{11}$$

We note, however, that we can generate a Bessel beam with a well-defined z component of angular momentum by preparing a close approximation of an eigenstate of $\hat{p}_\phi$ and allowing it to propagate [26].

We have shown that although the azimuthal momentum is not a true linear momentum, the azimuthal momentum density is a true density of linear momentum. Plane waves, which carry linear momentum with no azimuthal component, can therefore produce a vortex beam with nonzero total azimuthal momentum p_ϕ but an identically zero azimuthal component of linear momentum.

We thank Miles Padgett and Jörg Götte for helpful comments. This work was supported by the UK EPSRC and the Leverhulme Trust.

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